Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

21 May 2025

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.

1. [Maximum mark: 23]

This question asks you to use polynomial functions to model some situations in probability.

Two unbiased tetrahedral (four-sided) dice with faces labelled $1, 2, 3$ and 4 are thrown and the scores recorded.

The random variable M denotes the maximum of these two scores.

The probability distribution of M is given in the following table.

m	1	2	3	4
$\mathrm{P}(M=m)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

(a) Find $\mathrm{E}(M)$. [2]

An alternative way to represent the probability distribution of M is to use a polynomial function, G, where $G(t)=\sum_{m=1}^{4}\mathrm{P}(M=m)t^m$.

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An alternative way to represent the probability distribution of M is to use a polynomial function, G, where $G(t)=\sum_{m=1}^{4} P(M=m)t^m$.

Hence, for the distribution of M, $G(t)=\frac{1}{16}t+\frac{3}{16}t^2+\frac{5}{16}t^3+\frac{7}{16}t^4$.

(b) Find $G(1)$. [1]

(c) (i) Find $G'(t)$. [2]

(ii) Hence, show that $G'(1)=E(M)$. [3]

(This question continues on the following page)

(Question 1 continued)

A bag contains two red balls and three yellow balls.

Two balls are selected at random without replacement from the bag.

The random variable X denotes the total number of red balls selected.

The probability distribution of X can be represented by the polynomial function, G_X, where

$$G_X(t) = \sum_{x=0}^{2} \mathrm{P}(X = x)t^x .$$

(d) Show that $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$, making it clear how the coefficients of $G_X(t)$ have been determined. [5]

An unbiased coin and a biased coin are tossed.

The probability of obtaining a tail on the biased coin is p.

The random variable Y denotes the total number of tails obtained from tossing both coins.

The probability distribution of Y can be represented by the polynomial function, G_Y, where been determined. [5]

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The probability of obtaining a tail on the biased coin is p.

The random variable Y denotes the total number of tails obtained from tossing both coins.

The probability distribution of Y can be represented by the polynomial function, G_Y, where

$$G_Y(t) = \sum_{y=0}^{2} \mathrm{P}(Y = y)t^y.$$

(e) Given that the coefficient of t^2 in $G_Y(t)$ is $\frac{1}{3}$, find

(i) the value of p; [2]

(ii) an expression for $G_Y(t)$. [4]

The random variable Z denotes the sum of the total number of red balls selected, X, and the total number of tails obtained from tossing both coins, Y.

The probability distribution of Z can be represented by the function, G_Z, where

$$G_Y(t) = \sum_{y=0} P(Y = y)t^y .$$

(e) Given that the coefficient of t^2 in $G_Y(t)$ is $\frac{1}{3}$, find

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(ii) an expression for $G_Y(t)$. [4]

The random variable Z denotes the sum of the total number of red balls selected, X, and the total number of tails obtained from tossing both coins, Y.

The probability distribution of Z can be represented by the function, G_Z, where

$$G_Z(t) = G_X(t)G_Y(t) .$$

(f) For random variable Z, it can be shown that $G_Z'(1) = \mathrm{E}(Z)$.

Use this result to find $\mathrm{E}(Z)$. [4]

2. [Maximum mark: 32]

Informally, the curvature of a curve can be thought of as the amount by which the curve deviates from being a straight line. In this question, you will investigate the curvature of a variety of functions.

Consider any function f that can be differentiated twice.

The curvature, k, of any function f is defined by $k(x)=\frac{|f''(x)|}{\left(1+(f'(x))^2\right)^{\frac{3}{2}}}$.

Consider the family of linear functions $g(x)=mx+c$, where $x \in \mathbb{R}$ and m, $c \in \mathbb{R}$.

(a) Show that $k(x)=0$ for this family of linear functions. [2]

Consider the family of quadratic functions $h(x)=ax^2+bx+c$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and b, $c \in \mathbb{R}$.

For this family of quadratic functions, it is given that

$$k(x)=\frac{2|a|}{\left(1+(2ax+b)^2\right)^{\frac{3}{2}}} \text{ and}$$

The maximum value of $k(x)$ is denoted by $k_{\max}$.

(b) (i) By solving $k'(x) = 0$, find the value of x where $k_{\max}$ occurs.

You are **not** required to justify that this value of x leads to a maximum. [1]

(ii) Determine an expression for $k_{\max}$, in terms of a only. [2]

(iii) State the value of $\lim_{x \to \infty} k(x)$ and explain briefly the significance of this result. [2]

(iv) Consider the quadratic functions

$p(x) = -2x^2 + 2x - 10$ where $x \in \mathbb{R}$ and

$q(x) = 2x^2 + 5x + 25$ where $x \in \mathbb{R}$.

State which one of the following statements is true and justify your answer.

A. $k_{\max}$ of $p > k_{\max}$ of q

B. $k_{\max}$ of $p < k_{\max}$ of q

C. $k_{\max}$ of $p = k_{\max}$ of q [2]

(This question continues on the following page)

Consider the function $v(x) = \ln x$, where $x \in \mathbb{R}$, $x > 0$.

For v, it is given that

$$k(x) = \frac{x}{\left(1+x^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{1-2x^2}{\left(1+x^2\right)^{\frac{5}{2}}}.$$

(c) (i) Determine the exact value of x where $k_{\max}$ occurs.

You are **not** required to justify that this value of x leads to a maximum. [2]

(ii) Show that $k_{\max} = \dfrac{2\sqrt{3}}{9}$. [4]

Consider the function $w(x) = \mathrm{e}^x$, where $x \in \mathbb{R}$.

For w, it is given that

(ii) Show that $k_{\max} = \dfrac{2\sqrt{3}}{9}$. [4]

Consider the function $w(x) = \mathrm{e}^x$, where $x \in \mathbb{R}$.

For w, it is given that

$$k(x) = \frac{\mathrm{e}^x}{\left(1+\mathrm{e}^{2x}\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{\mathrm{e}^x\left(1-2\mathrm{e}^{2x}\right)}{\left(1+\mathrm{e}^{2x}\right)^{\frac{5}{2}}}.$$

(d) (i) Show that $k_{\max} = \dfrac{2\sqrt{3}}{9}$. [5]

(ii) Suggest a reason why v and w have the same $k_{\max}$. [1]

Consider a family of curves $y = \sqrt{r^2 - x^2}$, where $-r < x < r$, $y > 0$ and r is a positive constant.

$$k'(x) = \frac{e^x\left(1-2e^{2x}\right)}{\left(1+e^{2x}\right)^{\frac{5}{2}}}.$$

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Consider a family of curves $y = \sqrt{r^2 - x^2}$, where $-r < x < r$, $y > 0$ and r is a positive constant.

(e) (i) Show that $\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}$. [6]

(ii) Hence, show that the curvature, k, is constant for this family of curves. [5]